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1991 J. Phys. A: Math. Gen. 24 593

(http://iopscience.iop.org/0305-4470/24/3/017)

J. Phys. A: Math. Gen. 24 (1991) 593–601. Printed in the UK

Cyclic representations of a q-deformation of the Virasoro algebra

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Received 6 June 1990

Abstract. Finite-dimensional representations of a recently proposed q-deformation, $\mathcal{V}_q$, of the Virasoro algebra $\mathcal{V}_0$, for q a root of unity, are constructed. The representations have a cyclic structure, and only in some special cases are they simultaneously highest-weight representations of the SU(1, 1)-like subalgebra of $\mathcal{V}_q$. In the 'classical' limit only those cyclic representations that are related to regular similarity transformations that become singular in the limit $q \to 1$ reproduce highest-weight representations of the standard Virasoro algebra $\mathcal{V}_0$.

1. Introduction

The quantized universal enveloping algebras (or QUE algebras [1]) are 'quantum' deformations of Lie algebras; these deformations in terms of a parameter q (or $h \sim \ln q$) are subject to a correspondence principle stating that the quantized algebra yields a standard Lie algebra in the 'classical' limit $q \to 1$ (or $h \to 0$). Examples of QUE algebras have emerged in contexts like the quantum Yang–Baxter equation and conformal field theory (see [2–4] and references therein), and thereafter there has been intense exploration on the characterization and construction of such QUE algebras.

In the spirit of the suggestive work of Bernard and LeClair [5], a q-deformation has been recently proposed for the (centreless) Virasoro algebra $\mathcal{V}_0$ (see [6] and references therein; see also [7]). This q-deformation, $\mathcal{V}_q$, is quite natural from the viewpoint of q-commutators. Its SU(1, 1)-like subalgebra is related to the $SU_q(2)$ and $SU_q(1,1)$ QUE algebras proposed by Woronowicz [8] and Witten [9]—see also [10] for a 'physical' interpretation of $SU_q(1,1)$ in the context of quantum mechanics. It is not known, however, if $\mathcal{V}_q$ itself is a *bona fide* QUE algebra.

The aim of this paper is to provide representations of $\mathcal{V}_q$ in the special case in which the parameter q is a primitive root of unity (without loss of generality, we concentrate our attention on the case $q = e^{2\pi i/N}$, $N > 2$). In other q-deformed structures the representations for generic values of q are qualitatively the same as in the classical case; for the particular case in which q is a root of unity; however, the structure of the representations is a rather degenerate version of the classical one (see [2, 3, 11] for the case $SU_q(2)$). In our case, such a value of q will allow us to characterize $\mathcal{V}_q$ as a *cyclotomic* algebra, admitting cyclic N-dimensional representations. In some special cases these cyclic representations turn out to be at the same time highest-weight

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0305-4470/91/030593+09$03.50

representations of the subalgebra $\{L_0, L_{\pm 1}\}$ of $\mathcal{V}_q$. Only representations that are associated with similarity transformations that become singular in the limit $q \to 1$ give rise to highest-weight representations of the standard Virasoro algebra $\mathcal{V}_0$.

Finally, some considerations concerning the centred version of $\mathcal{V}_q$ are given, and we discuss some questions that remain still open in considering $\mathcal{V}_q$ as the definitive q-deformation of $\mathcal{V}_0$. We also obtain a new 'classical' algebra from $\mathcal{V}_q$.

2. A q-deformed Virasoro algebra

The operators L_n defining the Virasoro algebra $\mathcal{V}_0$ are indexed by integers that run over $\mathbb{Z}$, and they satisfy the commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \tag{2.1}$$

where the central charge c multiplies the centre of the algebra, identified with the identity operator. The q-deformed version $\mathcal{V}_q$ that has been recently proposed [6] is the following one:

$$q^{n-m}L_nL_m - q^{m-n}L_mL_n = [n-m]L_{n+m} \tag{2.2}$$

where the notation $[n]$ stands for the q-deformed quantity

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{2.3}$$

For our purposes it is convenient to introduce the concept of q-grading in this algebra by associating with the operator L_n a grading n. This grading, that we assume to be additive, can be extended to the whole algebra of operators, and we define the $*$-product of two operators A_n and A_m with definite grading n and m as the q-graded product

$$A_n * A_m \equiv q^{n-m}A_nA_m. \tag{2.4}$$

In terms of the q-graded or $*$-commutator

$$[A_n, A_m]_* = -[A_m, A_n]_* \equiv A_n * A_m - A_m * A_n \tag{2.5}$$

the algebra (2.2) becomes

$$[L_n, L_m]_* = [n-m]L_{n+m}. \tag{2.6}$$

This algebra is a centreless quantized version of (2.1). One of the problems in defining the corresponding q-centred algebra is that in this case the centre of the algebra $\mathcal{V}_q$ may not be identifiable with the identity operator, as we will see below. Moreover, since the $*$-product defined in (2.4) is not associative

$$(A_n * A_m) * A_p - A_n * (A_m * A_p) = q^{n-p}(q^n - q^{-p})A_nA_mA_p \tag{2.7}$$

the corresponding Jacobi identity for the $*$-commutator fails, and in principle there is no restriction on the structure constants accompanying the central terms.

In the case in which $q = e^{2\pi i/N}$ $(N > 2)$ the structure constants appearing in (2.6), as well as the gradation coefficients in the $*$-product, are invariant under translations $\mathcal{T}_N^k : n \to n + kN$ for arbitrary integer k. The group of translations $\mathcal{T}_N = \{\mathcal{T}_N^k, k \in \mathbb{Z}\}$ induces in $\mathcal{V}_q$ the algebra automorphism

$$L_n \to L_{n+kN} \qquad k = 0, \pm 1, \pm 2, \ldots \tag{2.8}$$

and the one-dimensional integer lattice labelling the operators splits into cells that may be referred by translations to a fundamental cell. This fundamental cell involves N generators that we choose to be the set $\{L_0, L_{\pm 1}, L_{\pm 2}, \ldots\}$; the 'last' operator in this set is $L_{N/2}$ for N even and $L_{\pm(N-1)/2}$ for N odd. This characterizes $\mathcal{V}_q$ as a cyclotomic algebra (see, for instance [12]), and the study of the infinite-dimensional algebra $\mathcal{V}_q$ can be reduced to the study of the finite-dimensional algebra $\hat{\mathcal{V}}_q = \mathcal{V}_q/\mathcal{T}_N$, where we have identified operators related by a transformation (2.8).

3. Representations of $\hat{\mathcal{V}}_q$

Our construction starts with the introduction of the two following $(N \times N)$ matrices $\mathbf{Q}$ and $\mathbf{H}$:

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & q & 0 & \ldots & 0 \\ 0 & 0 & q^2 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & q^{N-1} \end{pmatrix} \qquad \mathbf{H} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix} \tag{3.1}$$

that verify $\mathbf{Q}^N = \mathbf{H}^N = \mathbf{1}$ (these matrices are traditionally considered in other contexts, see [12, 13]). The set of matrices $\{\mathbf{Q}^n\mathbf{H}^m, n, m = 0, 1, 2, \ldots N-1\}$ span the N^2-dimensional linear space $\mathcal{M}$ of matrices $(N \times N)$; in particular any diagonal matrix can be expressed as a linear combination of powers of $\mathbf{Q}$. Since

$$\mathbf{HQ} = q\mathbf{QH} \qquad \mathbf{H}^n\mathbf{Q}^m = q^{nm}\mathbf{Q}^m\mathbf{H}^n. \tag{3.2}$$

$\mathbf{Q}$ induces in $\mathcal{M}$ the following natural gradation: a matrix M_n has a definite grading n if it verifies

$$\mathbf{M}_n\mathbf{Q} = q^n\mathbf{Q}\mathbf{M}_n. \tag{3.3}$$

Then $\mathbf{Q}^n$ and $\mathbf{H}^m$ have a grading 0 and m respectively, and any matrix $\mathbf{M}_n \in \mathcal{M}$ verifying (3.3) can be expressed uniquely as

$$\mathbf{M}_n = \left(\sum_{i=0}^{N-1} \alpha_i \mathbf{Q}^i\right)\mathbf{H}^n \tag{3.4}$$

i.e., as a diagonal matrix times $\mathbf{H}^n$. The quantities α_i are in general complex functions of the parameter q, $\alpha_i(q) = \alpha_i^0(q) + \mathrm{i}\alpha_i^1(q)$, where α_i^0 and α_i^1 are real functions of q. It should be stressed that, as it is usual in the context of QUE algebras and quantum groups [2, 3], the parameter q labelling our algebra is not complex-conjugated when we take the Hermitian conjugate of operators, since it is an 'external' quantity to the operator content of the algebra. Hence the complex conjugate of $\alpha_i(q)$ is given by $\bar{\alpha}_i(q) = \alpha_i^0(q) - \mathrm{i}\alpha_i^1(q)$, even if such functions are not invariant under $q \to q^{-1}$. This prescription should be understood in the sense that $\alpha_i(q)$ are functions on C' with values in C, where C' is the 'control parameter' space in which $q = \mathrm{e}^{2\pi\mathrm{i}/N}$ is defined; complex conjugation only affects C, while conjugation in C' amounts to the transformation $q \to q^{-1}$.

In this way we have endowed the space $\mathcal{M}$ with an additive and cyclic gradation under which it decomposes into a direct sum of N subspaces of definite grading:

$$\mathcal{M} = \bigoplus_{i=0}^{N-1} \mathcal{M}_i \qquad \mathcal{M}_i\mathcal{M}_j = \mathcal{M}_{i+j} \qquad i, j = 0, 1, \ldots N-1;\ (i+j) \bmod N. \tag{3.5}$$

This gradation is isomorphic to the one introduced in defining the $*$-commutator, and we have now a suitable framework to obtain representations for $\hat{\mathscr{V}}_q$. Some straightforward realizations of $\hat{\mathscr{V}}_q$ are for instance

$$l_n = \frac{a^n}{q-q^{-1}}\mathbf{H}^n \qquad l'_n = \frac{q^{bn^2}\mathbf{Q}^{2bn}}{q-q^{-1}}\mathbf{H}^n \tag{3.6}$$

for arbitrary quantities a and b; these operators verify the $*$-commutation relations (2.6), but they are rather trivial representations in the sense that they commute (in the standard sense), and they are uninteresting for our purposes.

To find non-trivial representations, let us first consider the operator L_0. Since L_0 has a grading 0, it can be expressed as

$$L_0 = \sum_{i=0}^{N-1} \alpha_i \mathbf{Q}^i \tag{3.7}$$

i.e. it is automatically a diagonal operator in our basis. On the other hand the n-graded operators L_n can be decomposed as a diagonal matrix times $\mathbf{H}^n$. Thus, in order for L_0 to verify the $*$-commutation relations (2.6), it is enough to impose the relation

$$[L_0, \mathbf{H}^n]_* = -[n]\mathbf{H}^n \tag{3.8}$$

which determines the diagonal elements of L_0, $a_p = \Sigma_{j=0}^{N-1} \alpha_j q^{jp}$, through the recurrence relation $q^{-n}a_p - q^n a_{p+n} = -[n]$, with the subindices taken mod N. The general solution is

$$a_p = \frac{1-q^{-2p}}{q-q^{-1}} + a_0 q^{-2p} \tag{3.9}$$

and we get for L_0 the general expression

$$L_0 = \frac{\mathbf{1}-\mathbf{Q}^{-2}}{q-q^{-1}} + a_0\mathbf{Q}^{-2} \tag{3.10}$$

where a_0 is a free parameter.

For L_n we consider now the following ansatz

$$L_n = \left(\frac{\mathbf{1}-\mathbf{Q}^{-2}}{q-q^{-1}} + F(n)\mathbf{Q}^{-2}\right)\mathbf{H}^n. \tag{3.11}$$

Substitution of these expressions in (2.6) determines the general solution for the functions $F(n)$:

$$F(n) = (a_0 - \beta) + \beta q^{-2n} \tag{3.12}$$

where β is a new free parameter. Finally we get the following realization for the $*$-commutator algebra (2.6):

$$L_n = \left(\frac{\mathbf{1}-\mathbf{Q}^{-2}}{q-q^{-1}} + (a_0 + \beta(q^{-2n}-1))\mathbf{Q}^{-2}\right)\mathbf{H}^n. \tag{3.13}$$

Since $\mathbf{Q}^\dagger = \mathbf{Q}$ and $\mathbf{H}^\dagger = \mathbf{H}^{-1}$, the Hermiticity condition $L_n^\dagger = L_{-n}$ is verified only if

$$a_0 - (\beta + \bar{\beta}) = \frac{1}{q-q^{-1}}. \tag{3.14}$$

The $*$-commutation relations (2.6) are invariant under similarity transformations $L_n \to \mathbf{A}L_n\mathbf{A}^{-1}$ defined by any regular matrix $\mathbf{A}$. Therefore, we can now obtain more general solutions from (3.13); the only constraint that we impose on the invertible matrices $\mathbf{A}$ is that they have to have a definite grading r in order to preserve the grading of the L_n's:

$$L'_n = \mathbf{A}_r L_n \mathbf{A}_r^{-1} \qquad \mathbf{A}_r = \mathbf{D}_r \mathbf{H}^r \qquad \mathbf{A}_r^{-1} = \mathbf{H}^{-r}\mathbf{D}_r^{-1} \tag{3.15}$$

(no summation in r is understood) where $\mathbf{D}_r$ is any regular diagonal matrix. In particular L_0 remains invariant, up to redefinitions of the free parameter a_0, under such similarity transformations.

Let us now consider the action of the algebra $\hat{\mathscr{V}}_q$ in the N-dimensional vectorial space $\mathscr{H}$ in which $\mathscr{M}$ acts, and choose the basis vectors of $\mathscr{H}$ as the eigenvectors of L_0 with the standard scalar product

$$|n\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1^{\leftarrow n} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \langle n|m\rangle = \delta_{n,m} \qquad n, m = 0, 1, \dots N-1. \tag{3.16}$$

The matrix elements of L_n are

$$\begin{aligned} L_n|m\rangle &= \gamma_n^m |m-n\rangle \qquad (m-n) \bmod N \\ \gamma_n^m &= \frac{1-q^{2(n-m)}}{q-q^{-1}} + a_0 q^{2(n-m)} + \beta q^{-2m}(1-q^{2n}). \end{aligned} \tag{3.17}$$

If we assume the Hermiticity condition (3.14), then $\bar{\gamma}_n^m = \gamma_{-n}^{m-n}$. The 'rotating' action of the operators L_n in this basis, see (3.17), characterizes the vectorial space $\mathscr{H}$ as a cyclic space with respect to the algebra $\hat{\mathscr{V}}_q$. Any of the vectors $|n\rangle$ can be considered as the generator of $\mathscr{H}$, and the basis has the structure of a Z_N-cyclic basis for $\mathscr{H}$. This is a consequence of the cyclotomic structure of the algebra $\hat{\mathscr{V}}_q$ which also entails that the operators $(L_n)^N$ are diagonal, and hence they trivially commute.

Each operator L_n has its conjugate operator L_{-n} in the $*$-commutator relations (2.6), and in this sense L_0 is a self-conjugate operator. For even values of N there is however another self-conjugate element in the operator algebra, namely $L_{N/2}$ that verifies

$$\begin{aligned} &[L_{N/2}, L_n]_* = [n]L_{n+N/2} \\ &[L_n, L_{n+N/2}]_* = [L_{n+N/2}, L_n] = 0. \end{aligned} \tag{3.18}$$

In particular L_0 and $L_{N/2}$ commute (in the standard sense), implying a degeneracy in the spectrum of L_0: the vectors $|n\rangle$ and $L_{N/2}|n\rangle \propto |n+N/2\rangle$ have the same L_0-eigenvalue, and $L_{N/2}$ induces this double degeneracy for even values of N.

In the case in which the free parameters a_0 and β take the particular values

$$a_0 = \frac{1}{q-q^{-1}}\left(1 - \frac{2q^{2p_0}}{1+q^2}\right) \qquad \beta = \frac{q^{2p_0+1}}{1-q^4} \tag{3.19}$$

for a given integer p_0, we get

$$\gamma_n^m = \frac{1}{q-q^{-1}}\left(1 - \frac{q^{2(p_0-m)}}{1+q^2}(1+q^{2n})\right). \tag{3.20}$$

For odd values of N we have $\gamma_1^{p_0} = \gamma_{-1}^{p_0-1} = 0$, i.e.

$$L_1|p_0\rangle = L_{-1}|p_0-1\rangle = 0 \tag{3.21}$$

and for even N, $\gamma_1^{p_0} = \gamma_1^{p_0+(N/2)} = \gamma_{-1}^{p_0-1} = \gamma_{-1}^{p_0-1+(N/2)} = 0$, i.e.

$$L_1|p_0\rangle = L_1|p_0+(N/2)\rangle = L_{-1}|p_0-1\rangle = L_{-1}|p_0-1+(N/2)\rangle = 0. \tag{3.22}$$

Hence for the special values (3.19) the cyclic representations of $\hat{\mathscr{V}}_q$ are simultaneously N-dimensional highest-weight representations of the subalgebra $\{L_0, L_{\pm1}\}$ of $\hat{\mathscr{V}}_q$ with highest-weight vector $|p_0\rangle$ and L_0-eigenvalue $q/(1+q^2)$. (For even values of N this representation is reducible, since it decomposes into two $(N/2)$-dimensional isomorphic highest-weight representations with highest-weight vectors $|p_0\rangle$ and $|p_0+(N/2)\rangle$.) Since (3.19) verify (3.14), we have $L_0^\dagger = L_0$, $L_{\pm1}^\dagger = L_{\mp1}$. Indeed, such highest-weight representations reproduce for $p_0 = 0$ the ones obtained by Fairlie [14] (see also [6]) for the QUE-algebras of Woronowicz and Witten (up to a renormalization of $L_{\pm1}$, and where the parameters r and s have to be replaced by $-q^{-1}$ and q^{-2}, respectively).

Under a similarity transformation (3.15) the matrix elements transform as

$$L'_n|m\rangle = \gamma_n'^m|m-n\rangle \qquad \gamma_n'^m = d_r^{-1}(m)d_r(m-n)\gamma_n^{m+r} \tag{3.23}$$

where $d_r(n)$ stands for the diagonal matrix elements of $\mathbf{D}_r$ (see (3.15)):

$$d_r(n) \equiv (\mathbf{D}_r)_{nn} = \frac{1}{(\mathbf{D}_r^{-1})_{nn}} \equiv \frac{1}{d_r^{-1}(n)} \tag{3.24}$$

and the quantities $(m-n)$ and $(m+r)$ are again understood mod N.

4. The 'classical' limit of $\hat{\mathscr{V}}_q$

The commutation relations (2.2) defining $\hat{\mathscr{V}}_q$ reproduce those of $\mathscr{V}_0$, (2.1), when the set of operators $\{L_n;\, |n| < \tilde{N}\}$ is considered in the limit $q \to 1$ $(h = 2\pi/N \to 0)$, and where $\tilde{N}$ is a function of N with the properties $\tilde{N} \to \infty$, $(\tilde{N}/N) \to 0$ as $N \to \infty$. In order to characterize the behaviour of the representations of $\hat{\mathscr{V}}_q$ in this classical limit, let us assume the following expansion for the free parameters a_0 and β:

$$a_0(q) \sim a_0(1) + \mathscr{O}(h) \qquad \beta(q) \sim \frac{1}{2\mathrm{i}h}\beta_0 + \mathscr{O}(1). \tag{4.1}$$

The asymptotic behaviour of the matrix elements γ_n^m for $|n| < \tilde{N}$ and around $m = 0$ is

$$\gamma_n^r \sim (a_0(1) + r - (1+\beta_0)n) + \mathscr{O}(h) \tag{4.2}$$

where $r = 0, \pm1, \pm2, \ldots$. In particular $\gamma_0^r \sim a_0(1) + r + \mathscr{O}(h)$ and the spectrum of the operator L_0 is unbounded above and below in the limit $q \to 1$. Therefore the cyclic representation (3.13) that we have considered gives rise in the classical limit to an unbounded representation of the standard Virasoro algebra $\mathscr{V}_0$.

How can we get highest-weight representations of $\mathscr{V}_0$? We know that for a generic (finite) value of N a similarity transformation leads to a qualitatively equivalent representation of $\hat{\mathscr{V}}_q$ (see (3.23)). This however turns out not to be true in the limit

$N \to \infty$, since similarity transformations that are regular for a generic value of N may become singular in the limit $N \to \infty$. The representations that are related through a similarity of this kind with our representation (3.13) give rise in the classical limit to qualitatively rather different representations. Indeed, to illustrate the fact that this kind of representations reproduces in the limit the relevant representations of the Virasoro algebra $\mathcal{V}_0$, let us consider the following particular case.

Consider the diagonal matrix

$$\mathbf{A} = \tfrac{1}{4}(\mathbf{Q}^{1/4} + \mathbf{Q}^{-1/4})^2 \tag{4.3}$$

whose action on the basis vectors is

$$\mathbf{A}|n\rangle = \tfrac{1}{4}(q^{n/4} + q^{-n/4})^2 |n\rangle \qquad 0 \leq n \leq N-1. \tag{4.4}$$

(The normalization of $\mathbf{A}$ is irrelevant; we have chosen it in such a way that $\mathbf{A}|0\rangle = |0\rangle$.) The eigenvalue of $\mathbf{A}$ in (4.4) is a function of period $2N$ that takes positive and decreasing values in the interval $[0, N-1]$; for generic values of N such elements are invertible, and the inverse matrix $\mathbf{A}^{-1}$ is well defined. Consider now the representation

$$\tilde{L}_n = \mathbf{A} L_n \mathbf{A}^{-1} \tag{4.5}$$

where L_n as in (3.13); the matrix elements are (see (3.23) and (3.24))

$$\tilde{\gamma}_n^m = \left(\frac{q^{(\lambda+m-n)/4} + q^{-(\lambda+m-n)/4}}{q^{m/4} + q^{-m/4}} \right)^2 \gamma_n^m$$

where

$$\lambda = \begin{cases} -N & \text{if } N \leq m-n \\ 0 & \text{if } 0 \leq m-n \leq N-1 \\ N & \text{if } m-n \leq -1. \end{cases} \tag{4.6}$$

Taking in these expressions the limit $N \to \infty$ for $|n|$, $m < \tilde{N}$ we get

$$\tilde{\gamma}_n^m = \begin{cases} a_0(1) + m - (1+\beta_0)n + \mathcal{O}(h) & \text{for } n \leq m \\ \mathcal{O}(h^2) & \text{for } n > m. \end{cases} \tag{4.7}$$

Therefore the representation (4.5) displays in the classical limit a highest-weight representation of the Virasoro algebra $\mathcal{V}_0$ with highest-weight vector $|0\rangle$ of 'conformal dimension' $h = a_0(1)$:

$$\begin{aligned} &\tilde{L}_0|0\rangle = h|0\rangle \\ &\tilde{L}_n|0\rangle = 0 \qquad n > 0 \\ &\tilde{L}_{-n}|0\rangle = (h + (1+\beta_0)n)|n\rangle \qquad n > 0 \\ &\tilde{L}_0|n\rangle = (h+n)|n\rangle \qquad n > 0 \end{aligned} \tag{4.8}$$

and the vectors $\{L_{-n}|0\rangle,\ n = 0, 1, 2, \ldots\}$ constitute a Verma module of $\mathcal{V}_0$. The action of $L_n(|n| < \tilde{N})$ on vectors $|N-r\rangle$, $r = 1, 2, \ldots$ is not well defined in the limit ($\tilde{\gamma}_n^{N-r}$ diverges for $n \leq -r$). However these vectors decouple from the Verma module in the classical limit, since they are not 'descendants' of $|0\rangle$.

In this way we have characterized highest-weight representations of $\mathcal{V}_0$ as the q-classical limit of cyclic representations of $\hat{\mathcal{V}}_q$. Finally, note that the Verma modules we have derived have quite a simple structure, since the descendant vector subspaces at a generic level n are one-dimensional (in other words, vectors like $(L_{-1})^2|0\rangle$ and $L_{-2}|0\rangle$ are linearly dependent).

5. Conclusions and outlook

The present work is just a step toward the construction and characterization of a q-deformation of the standard Virasoro algebra in the framework of the QUE algebras. There are still some open questions in considering $\mathscr{V}_q$ as a satisfactory deformation of $\mathscr{V}_0$.

One of them is how to define the centred version of $\mathscr{V}_q$. In section 2 we pointed out that the Jacobi identity for the $*$-commutator does not hold due to the non-associativity of the q-graded product $*$. This poses problems in constraining the structure constants accompanying the central terms in the $*$-commutator algebra. Moreover, the central term should $*$-commute with the oprators L_n; clearly the identity operator no longer plays this role, since for instance $\mathbf{1}/(q-q^{-1})$ in our construction verifies

$$\left[\frac{\mathbf{1}}{q-q^{-1}}, A_n\right]_* = -[n]A_n \tag{5.1}$$

for every operator A_n of grading n. The only operator in our cyclic representations that $*$-commutes with a generic A_n, and therefore with the operators L_n, is $\mathbf{\Omega}^{-2}$:

$$[\mathbf{\Omega}^{-2}, L_n]_* = 0. \tag{5.2}$$

Therefore, for the representations considered here, such an operator has to be identified with the centre of the $*$-commutator algebra.

Another open question is the comultiplication for $\mathscr{V}_q$. A coalgebra structure, which guarantees the existence of tensor products, should be shown to exist; otherwise such an algebra $\mathscr{V}_q$ would not constitute a Hopf algebra. However, and as another reminiscence of the lack of associativity of the product underlying the algebra $\mathscr{V}_q$, such a coproduct turns out to be rather involved [6, 14].

Finally, let us mention that there exists a rather different way of taking the limit $q\to 1$ of the commutator algebra (2.2) in the case $q=\mathrm{e}^{2\pi\mathrm{i}/N}$, namely the 'global' limit. Instead of considering the behaviour of the subset of operators $\{L_n; |n|<\tilde{N}\}$ in the limit $q\to 1$, we can consider the total set of redefined operators:

$$T_\theta \equiv (q-q^{-1})L_{N\theta/2\pi} \qquad \theta = 2\pi n/N \qquad n=0,1,\ldots N-1. \tag{5.3}$$

In the limit $N\to\infty$ the $*$-commutator algebra (2.6) now takes the form

$$\mathrm{e}^{\mathrm{i}(\theta-\theta')}T_\theta T_{\theta'} - \mathrm{e}^{\mathrm{i}(\theta'-\theta)}T_{\theta'}T_\theta = 2\mathrm{i}\sin(\theta-\theta')\,T_{\theta+\theta'} \tag{5.4}$$

where the angular variable θ runs continuously over the interval $[0, 2\pi]$. Apart from its trivial realization (the 'rotations' $T_\theta = \mathrm{e}^{\mathrm{i}\theta M}$ verify trivially the commutation relations (5.4)), this global classical limit of $\hat{\mathscr{V}}_q$ defines a generalized commutator algebra in the sense of Lepowsky and Wilson [15]. Indeed, putting $T_\theta = \Sigma_n T_n\,\mathrm{e}^{-\mathrm{i}n\theta}$ we get from (5.4)

$$T_{n+1}T_{m-1} - T_{m+1}T_{n-1} = (T_{n+1}\delta_{n,m-2} - T_{n-1}\delta_{n,m+2}) \tag{5.5}$$

that constitutes a particular case of the $\boldsymbol{Z}$-algebras considered in [15]. If we interpret T_θ as a field $T(\theta)$ defined on the unit circle, the generalized commutation relations (5.5) can be understood as a consequence of the non-trivial monodromy properties that the field $T(\theta)$ exhibits in (5.4).

References

[1] Drinfeld V 1985 *Sov. Math. Dokl.* **32** 254
Jimbo M 1985 *Lett. Math. Phys.* **10** 63; 1986 *Lett. Math. Phys.* **11** 247
[2] Alvarez-Gaumé L, Gómez C and Sierra G 1990 *Nucl. Phys.* B **330** 347
[3] Pasquier V and Saleur H 1990 *Nucl. Phys.* B **330** 523
[4] Majid S 1990 *Int. J. Mod. Phys.* A **5** 1
[5] Bernard D and LeClair A 1989 *Phys. Lett.* **227B** 417
[6] Curtright T L and Zachos C K 1990 *Phys. Lett.* **243B** 237
[7] Chaichian M, Kulish P and Lukierski J 1990 *Phys. Lett.* **237B** 401
[8] Woronowicz S 1987 *Publ. RIMS (Kyoto Univ.)* **23** 117
[9] Witten E 1990 *Nucl. Phys.* B **330** 285
[10] Narganes-Quijano F J 1990 The quantum deformation of SU(1, 1) as the dynamical symmetry of the anharmonic oscillator *Preprint* ULB-TH 90/02, Brussels University
[11] Roche P and Arnaudon D 1989 *Lett. Math. Phys.* **17** 295
[12] Fairlie D B and Zachos C K 1989 *Phys. Lett.* **224B** 101
[13] Weyl H 1950 *The Theory of Groups and Quantum Mechanics* (New York: Dover)
't Hooft G 1978 *Nucl. Phys.* B **138** 1; 1981 *Commun. Math. Phys.* **81** 267
Eguchi T and Nakayama R 1983 *Phys. Lett.* **122B** 59
Gehlen G, Rittenberg V and Ruegg H 1985 *J. Phys. A: Math. Gen.* **19** 107
Patera J and Zassenhaus H 1988 *J. Math. Phys.* **29** 665
[14] Fairlie D B 1990 *J. Phys. A: Math. Gen.* **23** L183
[15] Lepowsky J and Wilson R L 1984 *Invent. Math.* **77** 199